Aufgabe 3.1

(i)

$$f(n) = n(n-1)2^{n-2}$$

$$\Delta f(n) = (n+1)((n+1)-1)2^{(n+1)-2} - (n(n-1)2^{n-2})$$

$$= 2^{n-2} n(n+3)$$

(ii)

$$f(n) = n^5 3^{n(n-1)}$$

$$\Delta f(n) = (n+1)^5 3^{(n+1)((n+1)-1)} - (n^5 3^{n(n-1)})$$

$$= 3^{n^2} \left(3^n (n+1)^5 - 3^{-n} n^5\right)$$

(iii)

$$f(n) = \binom{n+1}{k+1}$$

$$\Delta f(n) = \binom{(n+1)+1}{k+1} - \binom{n+1}{k+1}$$

$$= -\frac{(n+1)!}{k!(-k+2+1)!}$$

(iv)

$$f(n) = \frac{n^4}{n^{\underline{4}}}$$

$$\Delta f(n) = \frac{(n+1)^4}{(n+1)^{\underline{4}}} - \left(\frac{n^4}{n^{\underline{4}}}\right)$$

$$= \frac{6n^2 + 8n + 3}{n(1-n)(n-2)(n-3)}$$

Aufgabe 3.2

(i) $s(n) = n^3$, $s(0) = 0$

(ii)

$$s(n) = an^4 + bn^3 + cn^2 + dn + e$$

$$\Delta s(n) = a(n+1)^4 + b(n+1)^3 + c(n+1)^2 + d(n+1) + e - (an^4 + bn^3 + cn^2 + dn + e)$$

$$= a(4n^3 + 6n^2 + 4n + 1) + b(3n^2 + 3n + 1) + c(2n+1) + d$$

Aus $a(4n^3 + 6n^2 + 4n + 1) + b(3n^2 + 3n + 1) + c(2n+1) + d = n^3$ folgt

$$a = \frac{1}{4}, b = -\frac{1}{2}, c = \frac{1}{4}, d = 0$$

Also: $s(n) = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 + e$

(iii) Setzt man für k die Werte $0, 1, 2, 3$ ein, erhält man:

- für $k = 0$: $j = 0$
- für $k = 1$: $h + j = 1 \Leftrightarrow h = 1$

- für $k=2$: $2\cdot g+2\cdot h+j=8 \Leftrightarrow g=3$
- für $k=3$: $6\cdot f+6\cdot g+3\cdot h+j=27 \Leftrightarrow f=1$

(iv)

$$\begin{aligned} k^3 &= fk^{\underline{3}}+gk^{\underline{2}}+hk^{\underline{1}}+j \\ &= fk(k-1)(k-2)+gk(k-1)+hk+j \\ &= f\left(k^3-3k^2+2k\right)+g\left(k^2-k\right)+hk+j \end{aligned}$$

Also gilt: $f=1, g=3, h=1, j=0$

(v)

$$\begin{aligned} k^3 &= \sum_{0\le i<3} \left\{ 3 \atop i \right\} p_i(k) \\ &= \left\{ 3 \atop 0 \right\} k^{\underline{0}} + \left\{ 3 \atop 1 \right\} k^{\underline{1}} + \left\{ 3 \atop 2 \right\} k^{\underline{2}} + \left\{ 3 \atop 3 \right\} k^{\underline{3}} \\ &= 0k^{\underline{0}}+1k^{\underline{1}}+3k^{\underline{2}}+1k^{\underline{3}} \end{aligned}$$

Also ergibt sich $f=1, g=3, h=1, j=0$

(vi)

$$\begin{aligned} s(n) &= \sum_{0\le i\le n} k^3 \\ &= \sum_{0\le i\le n} \left\{ 3 \atop i \right\} \cdot \frac{n^{\underline{i+1}}}{i+1} \\ &= \left\{ 3 \atop 0 \right\} \cdot \frac{n^{\underline{0+1}}}{0+1} + \left\{ 3 \atop 1 \right\} \cdot \frac{n^{\underline{1+1}}}{1+1} + \left\{ 3 \atop 2 \right\} \cdot \frac{n^{\underline{2+1}}}{2+1} + \left\{ 3 \atop 3 \right\} \cdot \frac{n^{\underline{3+1}}}{3+1} \\ &= j\cdot\frac{n^{\underline{1}}}{1}+h\cdot\frac{n^{\underline{2}}}{2}+g\cdot\frac{n^{\underline{3}}}{3}+f\cdot\frac{n^{\underline{4}}}{4} \\ &= 0\cdot\frac{n}{1}+1\cdot\frac{n(n-1)}{2}+3\cdot\frac{n(n-1)(n-2)}{3}+1\cdot\frac{n(n-1)(n-2)(n-3)}{4} \\ &= \frac{n^2(n-1)^2}{4} \\ &= \frac{n^4-2n^3+n^2}{4} \\ &= \frac{1}{4}n^4-\frac{1}{2}n^3+\frac{1}{4}n^2 \end{aligned}$$

(vii)
$$\left(\sum_{0\le k<n} k\right)^2 = \left(\frac{n(n-1)}{2}\right)^2 = \frac{n^2(n-1)^2}{4} = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$$

Also ist $\left(\sum_{0\le k<n} k\right)^2 = s(n)$

Aufgabe 3.3

(i)

$$\Delta f(n) = \frac{n\left(11(n+1)^2 + 48(n+1) + 49\right)}{6\left((n+1)+1\right)\left((n+1)+2\right)\left((n+1)+3\right)} - \frac{n\left(11n^2 + 48n + 49\right)}{6(n+1)(n+2)(n+3)}$$

$$= \frac{(n+1)\left(11n^2 + 70n + 108\right)}{6(n+2)(n+3)(n+4)} - \frac{n\left(11n^2 + 48n + 49\right)}{6(n+1)(n+2)(n+3)}$$

$$= \frac{(n+1)\left(11n^2 + 70n + 108\right)(n+1)}{6(n+1)(n+2)(n+3)(n+4)} - \frac{n\left(11n^2 + 48n + 49\right)(n+4)}{6(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{(n+1)\left(11n^2 + 70n + 108\right)(n+1) - n\left(11n^2 + 48n + 49\right)(n+4)}{6(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{18(n+2)(n+3)}{6(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{3}{(n+1)\cdot(n+4)}$$

$$= \frac{3}{n^2 + 5n + 4}$$

(ii)
$$\sum_{0\le k<n} \frac{3}{k^2 + 5k + 4} = f(n) - f(0) = \frac{n\left(11n^2 + 48n + 49\right)}{6(n+1)(n+2)(n+3)}$$

(iii) $$\frac{1}{k+1}-\frac{1}{k+4}=\frac{1(k+4)}{(k+1)(k+4)}-\frac{1(k+1)}{(k+4)(k+1)}$$
$$=\frac{1(k+4)-1(k+1)}{(k+1)(k+4)}$$
$$=\frac{k+4-k-1}{(k+1)(k+4)}$$
$$=\frac{3}{k^2+5k+4}$$

$$\sum_{0\le k<n}\frac{3}{k^2+5k+4}=\sum_{0\le k<n}\frac{1}{k+1}-\frac{1}{k+4}$$
$$=\sum_{0\le k<n}\frac{1}{k+1}+\sum_{0\le k<n}-\frac{1}{k+4}$$
$$=\sum_{1\le k<n+1}\frac{1}{k}-\sum_{4\le k<n+4}\frac{1}{k}$$
$$=\sum_{4\le k<n+1}\frac{1}{k}-\sum_{4\le k<n+1}\frac{1}{k}+1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}$$
$$=1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}$$
$$=\frac{n(11n^2+48n+49)}{6(n+1)(n+2)(n+3)}$$

Aufgabe 3.4

$(E^2-E-I)f=0,\ f(0)=1,\ f(1)=1$

$f(n+2)-f(n+1)-f(n)=0$

Ansatz für $f(n)=\lambda^n$

$E^2f(n)=\lambda^{n+2}=\lambda^2f(n)$

$Ef(n)=\lambda^{n+1}=\lambda f(n)$

$$\lambda^2f(n)-\lambda f(n)-f(n)=0$$
$$\Leftrightarrow(\lambda^2-\lambda-1)f(n)=0$$
$$\Leftrightarrow\lambda^2-\lambda-1=0$$
$$\Leftrightarrow\lambda_{1/2}^2=\frac{1}{2}\pm\frac{\sqrt{5}}{2}$$

$$f(n)=a(\lambda_1)^n+b(\lambda_2)^n$$
$$=a\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n+b\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n$$

Für $n=0$ ergibt sich $f(0)=a\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^0+b\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^0$

$$=a+b=1$$

Für $n=1$ ergibt sich $f(1)=a\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^1+b\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^1$

$$=a\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)+b\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)=1$$

Also ergibt sich:

$$a=\frac{1}{2}+\frac{\sqrt{5}}{10},b=\frac{1}{2}-\frac{\sqrt{5}}{10}$$

$$f(n)=\left(\frac{1}{2}+\frac{\sqrt{5}}{10}\right)(\lambda_1)^n+\left(\frac{1}{2}-\frac{\sqrt{5}}{10}\right)(\lambda_2)^n$$

Aufgabe 3.5

(i)

$$\Delta(f\cdot g)(n)=(\Delta f)(n)\,g(n+1)+f(n)(\Delta g)(n)$$
$$\Leftrightarrow \Delta\big(M(f,g)\big)(n)=M(\Delta f,Eg)+M(f,\Delta g)$$

(ii)

$$\begin{aligned}
\Delta\circ M &= M(I,\Delta)+M(\Delta,I)+M(\Delta,\Delta)\\
&= M(f,\Delta g)+M(\Delta f,g)+M(\Delta f,\Delta g)\\
&= f(n)\cdot\Delta g(n)+\Delta f(n)\cdot g(n)+\Delta f(n)\cdot\Delta g(n)\\
&= f(n)\cdot\big(g(n+1)-g(n)\big)+\big(f(n+1)-f(n)\big)\cdot g(n)+\big(f(n+1)-f(n)\big)\cdot\big(g(n+1)-g(n)\big)\\
&= f(n)\cdot g(n+1)-f(n)\cdot g(n)+f(n+1)\cdot g(n)-f(n)\cdot g(n)+f(n+1)\cdot g(n+1)-f(n+1)\\
&\quad \cdot g(n)-f(n)\cdot g(n+1)+f(n)\cdot g(n)\\
&= f(n+1)\cdot g(n+1)-f(n)\cdot g(n)\\
&= M(E,E)-M(f,g)\\
&= \Delta\circ M
\end{aligned}$$